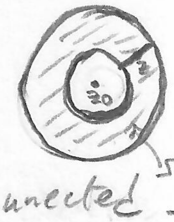
Weierstrass-Laurent theorem. Let $\Omega := \{z\in\mathbb{C} \mid r<|z-z_0|<R\}$ with $0\le r<R\le+\infty$, and let $f\in H(\Omega)$. Then, $\forall\ r<\rho<R$,

$$f(z)=\sum_{k\in\mathbb{Z}} a_k(z-z_0)^k\ ,\qquad a_k:=\frac{1}{2\pi i}\int_{\{|\zeta-z_0|=\rho\}^+}\frac{f(\zeta)}{(\zeta-z_0)^{k+1}}\,d\zeta\ ,\qquad (1)$$

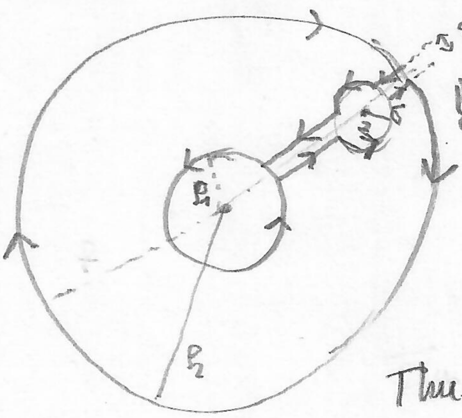
and the series $\sum_{k\ge0} a_k(z-z_0)^k$, $\sum_{k\ge1} a_{-k}(z-z_0)^{-k}$ converge absolutely in, respectively, $\{|z-z_0|<R\}$, $\{|z-z_0|>r\}$.



Ex. 1. Prove that $\forall z\in\Omega$, $\Omega' := \Omega\setminus\{z_0+t(z-z_0)\mid t\ge0\}$ is simply connected.

Ex. 2. Prove that the integral in (1) does not depend on $\rho\in(r,R)$ –

Proof. Up to a translation $(z\to z-z_0)$, we may assume that $z_0=0$. Fix $z\in\Omega=\{r<|z|<R\}$ and let $\Omega':=\Omega\setminus\{tz\mid t\ge0\}$ (which, by Ex 1 is simply connected). Fix $r<\rho_1<|z|<\rho_2<R$ and let, for $0<\sigma<\frac{R-r}{2}$ and $0<\varepsilon<\sigma$, $\Gamma_{\sigma,\varepsilon}$ denote the curve in the picture:



$\Gamma_{\sigma,\varepsilon}$ is a closed curve in Ω' hence

$$\frac{1}{2\pi i}\int_{\Gamma_{\sigma,\varepsilon}}\frac{f(\zeta)}{(\zeta-z)}\,d\zeta=0\qquad\forall\ 0<\varepsilon<\sigma\ .$$

When $\varepsilon\to0$, $\Gamma_{\sigma,\varepsilon}\to\Gamma_{\sigma,0}:=\Gamma_\sigma:=\{|\zeta|=\rho_1\}^+ + \{|\zeta-z|=\sigma\}^+ - \{|\zeta|=\rho_2\}^-$.

Thus,
$$0=\lim_{\varepsilon\to0}\int_{\Gamma_{\sigma,\varepsilon}}\frac{f(\zeta)}{\zeta-z}=\int_{\Gamma_\sigma}\frac{f(\zeta)}{\zeta-z}\qquad (2)$$

By Cauchy formula, $\frac{1}{2\pi i}\int_{\{|\zeta-z|=\sigma\}^+}\frac{f(\zeta)}{\zeta-z}=f(z)$, hence, by (2), we get

$$f(z)=f_2(z)+f_1(z)\ ,\quad\text{with } f_2(z):=\frac{1}{2\pi i}\int_{\{|\zeta|=\rho_2\}^+}\frac{f(\zeta)}{\zeta-z}\ ,\ f_1(z):=\frac{-1}{2\pi i}\int_{\{|\zeta|=\rho_1\}^+}\frac{f(\zeta)}{\zeta-z}\qquad (3)$$

Now, recalling that $|z|<\rho_2$, we find, since $|\zeta|=\rho_2$ on $\{|\zeta|=\rho_2\}^+$,

$$f_2(z)=\frac{1}{2\pi i}\oint_{|\zeta|=\rho_2}\frac{f(\zeta)}{\zeta\left(1-\frac{z}{\zeta}\right)}=\frac{1}{2\pi i}\sum_{k\ge0} z^k\oint_{|\zeta|=\rho_2}\frac{f(\zeta)}{\zeta^{k+1}}=:\sum_{k\ge0}a_kz^k$$

(geometric series for $\frac{1}{1-\frac{z}{\zeta}}$ and exchange of sum and integration)

While, in the integral defining f_1, we have $0<\frac{|\xi|}{|z|}=\frac{\rho_1}{|z|}<1$, so that,

$$f_1(z) = \frac{1}{2\pi i}\frac{1}{z}\int_{\{|\xi|=\rho_1\}^+} \frac{f(\xi)}{\left(1-\frac{\xi}{z}\right)} = \frac{1}{2\pi i}\,\frac{1}{z}\sum_{j=0}^{\infty} \int_{\{|\xi|=\rho_1\}^+} f(\xi)\left(\frac{\xi}{z}\right)^j d\xi$$

$$= \frac{1}{2\pi i}\sum_{j=0}^{\infty} \frac{1}{z^{j+1}}\int_{\{|\xi|=\rho_1\}^+} f(\xi)\,\xi^j \underset{k=-(j+1)}{=} \frac{1}{2\pi i}\sum_{k=-\infty}^{-1} z^k a_k . \blacksquare$$

Remark 1. The series expansions for f_1 and f_2, namely

$$f_2(z)=\sum_{k\geq 0} a_k(z-z_0)^k, \qquad f_1(z)=\sum_{k\leq -1} a_k(z-z_0)^k \qquad (4)$$

converge, respectively, for $|z-z_0|<R$ and $|z-z_0|>\frac{1}{r}$. This, by Cauchy-Hadamard formula, means that

$$R\leq \left(\overline{\lim_{k\to+\infty}}\,|a_k|^{\frac{1}{k}}\right)^{-1}, \qquad r\geq \overline{\lim_{k\to\infty}}\,|a_{-k}|^{\frac{1}{k}}. \qquad (5)$$

Ex.3 Prove that, if $\{a_k\}_{k\in\mathbb{Z}}$ are such that (5) holds, with $0\leq r<R\leq+\infty$, then $f(z):=\sum_{k\in\mathbb{Z}} a_k(z-z_0)^k$ defines a holomorphic function on $\{r<|z-z_0|<R\}$.

Remark 2 (i) The function f_1 in (4) has a removable singularity at ∞; in fact

$$\lim_{|z|\to+\infty} f_1(z) = \lim_{w\to 0}\sum_{k=1}^{\infty} a_k w^k = \sum_{k=1}^{\infty} a_k w^k\Big|_{w=0} = 0.$$

(ii) The representation of a function $f\in H(\Omega)$ as the sum of two functions, one analytic in $|z-z_0|<R$, and the other in $|z-z_0|>r$ (with removable singularity at ∞) is unique up to a constant: If $f_1+f_2=\tilde{f}_1+\tilde{f}_2$ with $\tilde{f}_1$ holomorphic in $|z-z_0|>r$ and removable singularity at ∞ and $\tilde{f}_2$ holomorphic in $|z-z_0|<R$ $(0\leq r<R\leq+\infty)$ then $\exists\, c\in\mathbb{C} \,|$

$$f_1=\tilde{f}_1+c \quad \text{and} \quad f_2=\tilde{f}_2-c.$$

Indeed, if $f_1+f_2=\tilde{f}_1+\tilde{f}_2 \Rightarrow F:=f_1-\tilde{f}_1=\tilde{f}_2-f_2$ on Ω, but, this implies that F is defined and holomorphic in $\{|z-z_0|<R\}$ and $\{|z-z_0|>r\}$, so it is an entire function with a removable singularity at ∞ (Hence $f_1-\tilde{f}_1$ is bdd at ∞) so that $F\equiv c$ by Liouville's theorem. Hence, $f_1=\tilde{f}_1+c$ and $f_2=\tilde{f}_2-c$. $\blacksquare$